Common Value Auction

Jacob Kohlhepp

October 29, 2021

Consider the following setting:

- There is one good and one owner who values the good at r. So owner's utility is $t(\theta)$ if the good is sold and r if not.
- There are n bidders who receive signals $\theta_i \sim U[\underline{\theta}, \bar{\theta}]$ prior to bidding.
- The bidders have common values, in that their utility from the good is: $\sum_i^n \theta_i/n$ less any transfers/bids.

1 Solution Using Mechanism Design

Intuitively, we should expect the owner to extract more surplus from the bidders, because he/she can rig the auction to reduce their information rents. First we write an expression for utility of individual i (ex-ante):

$$U_i(\theta, \tilde{\theta}) = E[p_i(\tilde{\theta}) \sum_i^n \theta_i/n|\theta_i] - E[t_i(\tilde{\theta}_i, \theta - i)]$$

Now write profit:

$$E[\sum_{i=1}^n t_i(\theta) + (1 - \sum_{i=1}^n p_i(\theta))r$$

Use the fact that profit will always be surplus minus utility:

$$E[\sum_{i=1}^n p_i(\theta)(\sum_i^n \theta_i/n - r) + r - \sum_{i=1}^n U_i(\theta)]$$

Note the owner's value appearing twice. Now we use the envelope condition on utility (using the fact that it is optimal to report truthfully):

$$\frac{dU(\theta_i)}{d\theta_i} = \frac{\partial U(\theta_i, \tilde{\theta}_i)}{d\theta_i}\bigg|_{\theta_i = \tilde{\theta}_i} = \frac{1}{n} E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})]$$

Now derive expected utility using integration by parts and the envelope result.

$$
\begin{aligned}
E[U_i(\theta)] &= \int_{\underline{\theta}}^{\bar{\theta}} U(\theta) f(\theta) d\theta \\
&= U(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} U'(\theta)(1 - F(\theta)) d\theta \\
&= U(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{n} E_{\theta_{-i}}[p_i(\theta_i, \theta_{-i})](1 - F(\theta_i)) d\theta_i \\
&= U(\underline{\theta}) + E\left[p_i(\theta_i, \theta_{-i}) \frac{1 - F(\theta_i)}{n f(\theta_i)}\right]
\end{aligned}
$$

I suppressed some conditioning notation in the utility functions. Plug this into profit:

$$
E[\sum_{i=1}^{n} \left\{ p_i(\theta)(\sum_{i}^{n} \theta_i/n - r) - p_i(\theta) \frac{1 - F(\theta_i)}{n f(\theta_i)} \right\} + r - nU(\underline{\theta})]
$$

Re-organize:[1]

$$
E[\sum_{i=1}^{n} \left\{ \frac{p_i(\theta)}{n} \left(\sum_{i}^{n} \theta_i - nr - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) \right\} + r - nU(\underline{\theta})]
$$

Note that if $\frac{1-F(\theta_i)}{f(\theta_i)}$ is decreasing and $\sum_i^n \theta_i - nr - \frac{1-F(\theta_i)}{f(\theta_i)}$ for the maximum reported value exceeds 0, we allocate the good to the highest value bidder. If it does not exceed 0, we keep the good. If the MR function is decreasing we randomly allocate the good. If it is crazy non-monotone we need to iron. In this case we have specified that it is uniform, so we have that:

$$
MR(\theta) = \frac{1 - (\theta - \underline{\theta})/(\bar{\theta} - \underline{\theta})}{1/(\bar{\theta} - \underline{\theta})} = \bar{\theta} - \theta
$$

This is clearly decreasing as supposed. The explicit allocation function is:

$$
p_i(\theta) = \begin{cases} 1 \text{ if } \theta_i > \max\{\theta_j\}_{j=1}^n \text{ and } \sum_i^n \theta_i - nr - \frac{1-F(\theta_i)}{f(\theta_i)} \geq 0 \\ 0 \text{ else} \end{cases}
$$

Let us now derive bidding strategies for a first-price auction. To do so, notice that in a first-price auction utility is $\bar{v} - b$ if you win (where $\bar{v}(\theta_i)$ is the expected average of all signals given i's signal). Suppose we wish to find a symmetric BNE with bidding functions denoted $b(\theta_i)$. We can do the utility trick, and equate the game utility with the utility of an incentive compatible mechanism:

$$
\begin{aligned}
(\bar{v}(\theta_i) - b(\theta_i)) Pr(\theta_i > \max\{\theta_j\}_{j=1}^n | \theta_i) &= U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta_i} \frac{1}{n} E_{\theta_{-i}}[p_i(s, \theta_{-i})] ds \\
&= \frac{1}{n} \int_{\underline{\theta}}^{\theta_i} Pr(s \geq \max\{\theta_j\}_{j=1}^n\}) ds \\
&= \frac{1}{n} \int_{\underline{\theta}}^{\theta_i} \left(\frac{s - \underline{\theta}}{\bar{\theta} - \underline{\theta}} \right)^{n-1} ds
\end{aligned}
$$

[1] Notice that the MR expression is not the same as the one for IID private values.

where I use that the bottom type wins with probability 0, and the optimal mechanism sets $p_i(\theta) = \mathbb{I}\{\theta_i > \max\{\theta_j\}_{j=1}^n\}$ and some uniform tricks. Solving for the bid function gives us:

$$b(\theta) = \frac{n-1}{n}E[\theta] + \frac{\theta_i}{n} - \left[\frac{\theta_i - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right]^{-(n-1)} \frac{1}{n} \int_{\underline{\theta}}^{\theta_i} \left(\frac{s - \underline{\theta}}{\bar{\theta} - \underline{\theta}}\right)^{n-1} ds$$

where I assumed r was below the support of θ so I could ignore it. This simplifies a bit further:

$$b(\theta) = \frac{n-1}{n}E[\theta] + \frac{\theta_i}{n} - \frac{1}{n^2}\frac{\theta_i - \underline{\theta}}{\bar{\theta} - \underline{\theta}}$$

With standard uniform, we get a very nice solution:

$$b(\theta) = \frac{n-1}{2n} + \frac{\theta_i}{n} - \frac{1}{n^2}\theta_i$$

With two bidders, this gives the bid function $b(\theta_i) = 1/4 + \theta_i/4$. For 3, it gives $b(\theta_i) = 1/9 + 2/9\theta_i$. For i.i.d. values, FPA bids are just $\theta_i \frac{N-1}{N}$. Clearly, for 2 bidders we have more aggressive bidding, in the sense that all types bid more when values are common. This is less true for 3 bidders.

In the limit, bids functions converge to a constant 1/2, which is quite different than the independent values case, where the FPA bid function converges to $b(\theta_i) = \theta_i$, i.e. bid your value. This has the further effect that profit in the limit with common values is 1/2, but it is 1 in private values. Notice though that in both cases, the firm is capturing nearly all surplus in the limit, because maximum surplus is 1/2 for the common values and 1 for i.i.d. values.

Another thing to note is that we still have a form of revenue equivalence in this setting, that is any mechanism which gives the good to the highest value bidder will maximize revenue. This result DOES NOT hold for general models of interdependent values, where bidders may have arbitrarily correlated values. In general, it is often the case that revenue equivalence will fail, and second price auctions will outperform first-price auctions and generate more revenue. See Prof. Sadzik's 2nd year theory class if you want to learn more about this.