Notes on Hamiltonians

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1 Hamiltonians (aka HJBs or Optimal Control): A Brief Tutorial

Hamiltonians are a way of deriving optimal solutions when the object being maximized over is a function. The most common example is time. In continuous time macro-type models, the consumption and capital accumulation rules are equations that are functions of t. In this class, we will be using HJBs to solve problems where the solution is a function of types, θ. The techniques are the same in both examples, just know that if you look this up online most problems will be dealing with time as the main object rather than types.

In general, we setup the HJB, take FOC's, use complementary slackness, and then get some solutions. When the problem satisfies some conditions of differentiability, and when the endpoint is unconstrained, the conditions we derive are necessary and sufficient for the solution. This is called the Maximum Principle. I will not review the HJB theorems, but they exist. They are like the K-T Theorem for Lagrangian's. A good reference for a heuristic derivation and how-to-guide is Sala-i-Martin's book, "Economic Growth", Appendix Section A.3.

1.1 Example Setup

We can use HJBs to solve problems that look like this[1]:

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \theta q(\theta) - c(q(\theta)) - U(\theta) dF(\theta)$$

s.t.

$$U'(\theta) = q(\theta) \tag{ICFOC}$$

$$U(\underline{\theta}) = 0 \tag{IR}$$

where notice that we have gone straight to the ICFOC. In the homework, you will need to derive this first. In general, writing the problem in a way that can be solved with HJBs is not a trivial task.

Comments:

- For these problems we think of the control as the quantity, that is $q(\theta)$ is the control function we are trying to optimize.
- U can be thought of as the state variable. It transitions according to the ICFOC.
- The IR condition is what is referred to as a boundary condition. We need it to pin down where U starts. In macro problems this is usually something like initial capital.

[1] This is from page 13 of Prof. Board's notes on Asymmetric Information.

1.2 Create Hamiltonian

Similar to the Lagrangian, we setup an unconstrained problem which has multipliers:

$$\mathcal{H} = [\theta q(\theta) - c(q(\theta)) - U(\theta)]f(\theta) + \mu(\theta)q(\theta)$$

Note we are implicitly doing point-wise maximization, since we have dropped the integral.

1.3 Take FOCs

We take derivatives w.r.t. q, the state, and set it to 0:

$$\frac{\partial H}{\partial q} = 0$$

Then take the derivative w.r.t. U the state, and set it equal to the negative of the derivative of the multiplier:

$$\frac{\partial H}{\partial U} = -\frac{d\mu}{d\theta}$$

We also have a complementary slackness condition, which for finite bounds of θ is: $\mu(\bar{\theta})U(\bar{\theta}) = 0$. This means the highest type gets no utility or the constraint is slack. In general there will be positive utility for the highest type, so the constraint will be slack.
The above FOCs give:

$$[\theta - c'(q(\theta))]f(\theta) + \mu(\theta) = 0 \text{ and } f(\theta) = \frac{d\mu}{d\theta}$$

1.4 Manipulate FOC's and Slackness to Get Answers

This step is now problem specific. One technique that will help us get more results is to integrate the state variable FOC:

$$\mu(\theta) = \int_{\theta}^{\bar{\theta}} f(s)ds + \mu(\bar{\theta}) = 1 - F(\theta)$$

Notice that we guess $\mu(\bar{t}heta) = 0$ from complementary slackness because we know that the highest type generally gets positive utility (this is related to no distortion at the top).
Now we can use the control FOC with this substituted:

$$\theta - \frac{1 - F(\theta)}{f(\theta)} = c'(q(\theta))$$

which is the result we got in class.

2 Example: Spence-Like Problem

I will now do one more example. Consider an education choice model similar to Spence, where agents with type θ have utility given by:

$$U(e, \theta) = -e/\theta + w(e)$$

and have outside option 0. So utility depends on wage, w and the cost of education, which is lower for higher types. Now suppose firm utility from hiring agent of type θ is:

$$\Pi(e) = \theta log(e) - w(e)$$

So now education is not useless, it increases profit, as does ability. Note that in the first-best benchmark, surplus maximizing education would be $e = \theta^2$. To setup the problem in an HJB friendly way, consider a direct-revelation mechanism which from report θ assigns education and wage (transfer). That is our new problem is:

$$\max_{w(\theta),e(\theta)} E[\theta log(e(\theta)) - w(\theta)]$$

s.t.

$$w(\theta) - e(\theta)/\theta \geq w(\tilde{\theta}) - e(\tilde{\theta})/\theta \forall \theta, \tilde{\theta} \tag{IC}$$

$$w(\theta) - e(\theta)/\theta \geq 0 \forall \theta \tag{IR}$$

Normally we would simplify this in the typical way we learned this week. But we can do it another way with HJBs. To do so, first we can employ the envelope condition trick from lecture:

$$\frac{dU(\theta)}{d\theta} = \frac{\partial U(\theta, \tilde{\theta})}{\partial \theta}\bigg|_{\tilde{\theta}=\theta} = \frac{e(\theta)}{\theta^2}$$

This is our law of motion for the control. Now re-write profit as surplus minus utility and are ready for HJBs:

$$\max_{e(\theta)} E[\theta log(e(\theta)) - e(\theta)/\theta - U(\theta)]$$

s.t.

$$\frac{dU(\theta)}{d\theta} = \frac{e(\theta)}{\theta^2}$$

Our HJB is:

$$\mathcal{H} = f(\theta)[\theta log(e(\theta)) - e(\theta)/\theta - U(\theta)] + \mu(\theta)\frac{e(\theta)}{\theta^2}$$

Taking FOCs:

$$\frac{\partial H}{\partial U} = -f(\theta) = -\frac{\partial \lambda}{\partial \theta}$$

$$\frac{\partial H}{\partial e} = f(\theta)\left(\frac{\theta}{e(\theta)} - \frac{1}{\theta}\right) + \mu(\theta)\frac{1}{\theta^2} = 0$$

Slackness: $\mu(\bar{\theta})U(\bar{\theta}) = 0$.

Integrate the state variable FOC like before using that $\mu(\theta) = 0$, meaning the top type has no desire to copy someone else:

$$\mu(\theta) = 1 - F(\theta)$$

Plug this into the other FOC to get our final expression:

$$f(\theta)\left(\frac{\theta}{e(\theta)} - \frac{1}{\theta}\right) - \frac{1-F(\theta)}{\theta^2} = 0$$

$$\leftrightarrow \frac{\theta}{e(\theta)} - \frac{1}{\theta} + \frac{1-F(\theta)}{f(\theta)\theta^2} = 0$$

$$\leftrightarrow \left(\frac{1}{\theta^2} - \frac{1-F(\theta)}{f(\theta)\theta^3}\right)^{-1} = e(\theta)$$

To get wages, note that:

$$U(\theta) = \int_{\underline{\theta}}^{\theta} \frac{dU(\theta)}{d\theta} = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{e(\theta)}{\theta^2} d\theta$$

And that:

$$U(\theta) = w(\theta) - e(\theta)/\theta$$

So:

$$w(\theta) = \int_{\underline{\theta}}^{\theta} \frac{e(\theta)}{\theta^2} d\theta + e(\theta)/\theta$$

which simplifies after plugging in optimal education:

$$w(\theta) = \int_{\underline{\theta}}^{\theta} \left(1 - \frac{1-F(\theta)}{f(\theta)\theta}\right)^{-1} d\theta + \left(\frac{1}{\theta} - \frac{1-F(\theta)}{f(\theta)\theta^2}\right)^{-1}$$

It remains to be seen whether I satisfied all the conditions necessary so that these are indeed the maximizers. But if regularity conditions hold, this is the optimal education and wage schedule.
If we were to change this so that profit was just $log(e)$ not $\theta log(e)$ we would get the nicer:

$$e(\theta) = \frac{\theta^2}{\theta - (1-F(\theta))/f(\theta)}$$

The bottom has the normal marginal revenue expression. If MR is decreasing, then we see that education is monotone increasing.