Bayesian vs. Dominant Strategy Implementation

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1 Comment on Pivot Losing Money

Consider a pivot mechanism, defined as a *VCG mechanism* with a specific transfer function:

$$t_i(\tilde{\theta}) = \left[\sum_{j \neq i} \theta_j - c\right][p^*_{-i}(\tilde{\theta}_{-i}) - p^*(\tilde{\theta})]$$

We consider the case where there are two agents, and their θ are such that:

$$\theta_1, \theta_2 < c < \theta_1 + \theta_2$$

In this case does the pivot mechanism lose money? Yes. To see why recall:

- In this specific case, both people are pivotal: the project is not constructed unless both participate.
- Do not forget that pivot is just a special case of VCG, so the allocation function is still:

  $$p^*(\tilde{\theta}) = 1 \text{ if } \sum_i \theta_i > c$$

  This helps in showing that pivot loses money.
- When we talk about pivot and VCG we are considering *dominant strategy implementation*, a harder form of implementation than bayesian. This is why we work with $\tilde{\theta}$ rather than the actual values, because we consider what people will do even when they believe others are reporting non-truthfully.

Back to the problem. To show pivot loses money, we need to evaluate:

$$t_1(\theta) + t_2(\theta) - c$$

We subtract c because the project cost is c, and the firm has to pay it. We evaluate at θ because we know that in equilibrium everyone plays their true value (since it is dominant to do so). Recall that with our assumption, $p^*(\theta) = 1$ and $p^*_{-i}(\theta_{-i}) = 0$. Plugging this into the transfer functions gives that:

$$\pi = -(\theta_2 - c) + -(\theta_1 - c) - c = -(\theta_1 + \theta_2) + c < 0$$

where the last inequality is from our original assumption. If you were to return to a situation with random values, this would generally mean there are realizations for which the firm loses money.

2 Bayesian vs Dominant Strategy Implementation Example

2.1 The Context

Consider the following environment:

1. There is a single public good (think of it as a community park) which costs $0 < c < 1$ to build and 2 members of a neighborhood.

2. The neighbors each privately value the good at θ_i, which is i.i.d. and uniform from $U[0, 1]$.

3. They wish to create an organization to determine whether to build the swimming pool, and collect contributions.

2.2 The Questions

1. Suppose the neighbors want to implement the park in dominant strategies, where everyone's individual rationality constraint is satisfied (assume people must receive weakly positive utility). How can the efficient outcome be achieved?

2. If it can be achieved, can we guarantee budget balance?

3. If not, what is expected budget deficit or surplus?

4. Suppose we wish to implement the park using Bayesian Implementation. How can we achieve efficiency?

5. Can we guarantee budget balance? If not, what is the expected deficit/surplus?

2.3 Solution

1. Suppose the neighbors want to implement the park in dominant strategies, where everyone's individual rationality constraint is satisfied (assume 0 outside option). How can the efficient outcome be achieved?

   We have a proposition that says that dominant strategy implementation can always be achieved with VCG, and all others ways are also VCG. So we suggest VCG:

   $$p^*(\tilde{\theta}) = \mathbb{I}\{\tilde{\theta}_1 + \tilde{\theta}_2 > c\}$$

   $$p^*_{-i}(\tilde{\theta}_{-i}) = \mathbb{I}\{\tilde{\theta}_{-i} > c\}$$

   We need the mechanism to be individually rational, so we also suggest Pivot:

   $$t_i(\tilde{\theta}) = (\tilde{\theta}_{-i} - c)(p^*_{-i}(\tilde{\theta}_{-i}) - p^*(\tilde{\theta}))$$

   This implements the efficient outcome, i.e. build only if cost is less than benefit.

2. If it can be achieved, can we guarantee budget balance?

   We can think of this as asking, can we assure transfers are greater than cost with probability 1? With this in mind, we cannot guarantee budget balance if $0 < c < 2$. To see why, note that sum of transfers less cost is:

$$\pi = \begin{cases} -(\theta_1+\theta_2)+c \text{ if } \theta_1,\theta_2 < c < \theta_1+\theta_2 \\ -\theta_2 \text{ if } \theta_2 < c < \theta_1 \\ -\theta_1 \text{ if } \theta_1 < c < \theta_2 \\ -c \text{ if} c < \theta_1,\theta_2 \end{cases}$$

Notice we never make money. This is partly because we need to give everyone positive utility: we need individual rationality. If we weaken individual rationality to say that some people can be forced to pay money, we can achieve budget balance.

The general result is that we cannot have efficiency, dominant strategy implementation, and individual rationality all at the same time. We need to give up one.

3. If not, what is expected budget deficit or surplus?

   I do not do this, but the way to do it is straightforwards (just cumbersome). We need to consider all the cases above.

4. Suppose we wish to implement the park using Bayesian Implementation. How can we achieve efficiency? In Bayesian implementation we can use our usual machine to find the answer. First, note that welfare is:

$$W = E[p(\theta)(\theta_1+\theta_2-c)$$

We clearly must build if $\theta_1+\theta_2 \geq c$ to achieve efficiency. But then our propositions tell us that utility must be:

$$U_1(\theta) = \int_{\underline{\theta}}^{\theta} E_{-i}[p(s,\theta_2)]ds$$

with U_2 symmetric. Then transfers must be:

$$t_1(\theta) = E_{-1}[p(s,\theta_2)]\theta_1 - \int_{\underline{\theta}}^{\theta} E_{-1}[p(s,\theta_2)]ds$$

Plugging in p:

$$t_1(\theta) = E_{-1}[\mathbb{I}\{\theta_1+\theta_2 \geq c\}]\theta_1 - \int_{\underline{\theta}}^{\theta} E_{-i}[\mathbb{I}\{s+\theta_2 \geq c\}]ds$$

To go farther, note that:

$$E_{-1}[\mathbb{I}\{\theta_1+\theta_2 \geq c\}] = min\{1-c+\theta_1, 1\}$$

$$t_1(\theta) = min\{1-c+\theta_1,1\}\theta_1 - \int_{\underline{\theta}}^{\theta} min\{1-c+s,1\}ds$$

$$t_1(\theta) = \begin{cases} \theta_1 - \int_{\underline{\theta}}^{c} 1-c+sds - \int_c^1 1ds \text{ if } \theta_1 > c \\ (1-c+\theta_1)\theta_1 - \int_{\underline{\theta}}^{\theta} 1-c+sds \text{ else} \end{cases}$$

$$t_1(\theta) = \begin{cases} \theta_1 - \int_{\underline{\theta}}^{c} 1-c+sds - \int_c^1 1ds \text{ if } \theta_1 > c \\ (1-c+\theta_1)\theta_1 - \int_{\underline{\theta}}^{\theta} 1-c+sds \text{ else} \end{cases}$$

Consider $c = 0.5$. Then:

$$t_1(\theta) = \begin{cases} min\{0.5 + \theta_1, 1\}\theta_1 - 0.875 \text{ if } \theta_1 > 0.5 \\ min\{1 - c + \theta_1, 1\}\theta_1 - \int_{\underline{\theta}}^{\theta} min\{1 - c + s, 1\}ds \text{ else} \end{cases}$$

5. Can we guarantee budget balance? If not, what is the expected deficit/surplus?

Side note: if we hired a firm to do this, they would solve the usual problem under Bayesian implementation and get:

$$\Pi = E[\sum_{i=1}^{2}(\frac{1 - F(\theta_i)}{f(\theta_i)} - c)p(\theta)] + U(\underline{\theta}, \underline{\theta})$$

where notice that p DOES NOT depend on i, because we cannot exclude people from a public good. We can write this nicer by pulling out p and using our uniform assumption:

$$\Pi = E[p(\theta)\Bigg(\sum_{i=1}^{2}(\theta - c)\Bigg)] + U(\underline{\theta}, \underline{\theta})$$

We can see here that we want to build if the summation is positive, that is if $\theta_1 + \theta_2 - 2c > 0$. This is our build decision. Implicitly, since a uniform build decision is by default monotone, we have satisfied both IC and IR for everyone.