Review of Monotone Comparative Statics

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1 Quick Review of Monotone Comparative Statics

Big Idea: These concepts help us say things about how the solution of economic models changes when a certain parameter rises or falls.

1.1 The Older Way: Implicit Function Theorem

Suppose we have the following (completely abstract for now) economic maximization problem:[1]

$$\max_{x \in X} b(x, \theta) - c(x)$$

where suppose all the functions satisfy conditions so the solution is the FOC:

$$b_x(x, \theta) = c'(x)$$

Then we know that the solution is implicitly defined as a function of θ:

$$b_x(x^*(\theta), \theta) = c'(x^*(\theta))$$

If both functions are twice continuously differentiable and we have a rank condition ($b_{xx}(x^*(\theta), \theta) \neq c''(x^*(\theta))$)We can appeal to the **Implicit Function Theorem** to say that $x^*(\theta)$ is also continuously differentiable, and the derivative is:

$$\frac{dx^*(\theta)}{d\theta} = \frac{b_{x\theta}(x^*(\theta), \theta)}{c''(x^*(\theta) - b_{xx}(x^*(\theta), \theta)}$$

If we have concavity of b in x, and convexity of c, and $b_{x\theta} > 0$ then this means that locally, the solution is increasing in θ. If the above captures production, with θ being perhaps price, then this makes sense.

1.1.1 Issues

- We needed continuity and derivatives everywhere.
- We needed conditions on both functions and even cross-derivatives.
- the Cross-derivative property, $b_{x\theta}$¿0, implies some sense of complementary between the two things. We want more x when θ is higher.

[1] Source are Wolitzky notes: `https://ocw.mit.edu/courses/economics/14-121-microeconomic-theory-i-fall-2015/lecture-slides/MIT14_121F15_4S.pdf`

1.2 The Newer Way: Monotone Comparative Statics

We wish to do what we did in the last section, but with less assumptions. I will introduce some new notions of "complementary" that do not need derivatives. Then I will introduce some theorem that apply these ideas to get some sort of comparative static. Main source is Topkis (1998) book and the lectures notes from Wolitzky notes referenced before.

1.2.1 What does it mean to be increasing?

With scalar solutions, it is clear that increasing just means greater. But with sets of solutions, what does it mean to increase? Topkis and all the literature use this definition:

Definition 1. *A set $A \subseteq \mathcal{R}$ is greater than set $B \subseteq \mathcal{R}$ in the* ***strong set order*** *if, for any $a \in A$ and $b \in B$:*

$$\max\{a, b\} \in A \text{ and } \min\{a, b\} \in B$$

so our sense of increasing is this. Note that when sets are just singletons, this reduces to just our normal idea of being bigger.

1.2.2 Increasing Differences

Definition 2. *We say a function $f : X \times \Theta \to \mathcal{R}$ has* ***increasing-differences*** *in (x, θ) if, for any $x^H \geq x^L$ and $\theta^H \geq \theta^L$:*

$$f(x^H, \theta^H) - f(x^L, \theta^H) \geq f(x^H, \theta^L) - f(x^L, \theta^L)$$

- Intuition: The benefit of higher x over lower x does not go down as θ goes up.
- This is equivalent to $\frac{\partial^2 f(x,\theta)}{\partial x \partial \theta} \geq 0$ with differentiability, what we needed in the implicit function method.

Not very helpful theorem:

Theorem 1. *If f has increasing differences in x, θ then $X^*(\theta)$ is increasing in the* ***strong set order.***

1.2.3 Single-crossing

We will use this a little in this class:

Definition 3. *We say a function $f : X \times \Theta \to \mathcal{R}$ is* ***single-crossing*** *in (x, θ) if, for any $x^H \geq x^L$ and $\theta^H \geq \theta^L$:*

$$f(x^H, \theta^L) \geq f(x^L, \theta^L) \implies f(x^H, \theta^H) \geq f(x^L, \theta^H)$$

and

$$f(x^H, \theta^L) > f(x^L, \theta^L) \implies f(x^H, \theta^H) > f(x^L, \theta^H)$$

- Intuition: If it is better to choose x^H under low θ, it is also better to choose x^H under high θ.
- useful to think about the single-variable case, where this basically says that a function crosses the x-axis at most once, and from below. See Wikipedia page for single-crossing.

Theorem 2. *(Milgrom and Shannon Monotonicity) If f is* ***single-crossing*** *in (x, θ) then $X^*(\theta)$ is increasing in the strong set order.*

1.2.4 Supermodularity

When x, θ are vectors, increasing differences is not enough. We could have that components with x impact each other, so as the Wolitzky notes put it, we need "complementarity between the x components too!

Definition 4. *We say a function $f : X \times \Theta \to \mathcal{R}$ is **supermodular** in x if, for all $x, y \in X$ and $\theta \in \Theta$:*

$$f(x \vee y, \theta) - f(x, \theta) \geq f(y, \theta) - f(x \wedge y, \theta)$$

Comments:

- Recall that x, y are vectors. The new notation $\vee$ is the component-wise maximum and $\wedge$ is the component-wise minimum.
- The twice continuously differentiable version of supermodularity is:

$$\frac{\partial^2 f(x,\theta)}{\partial x_i \partial x_j} \geq 0 \forall x \in X, \theta \in \Theta, i \neq j$$

Theorem 3. *(Topkis) If $X \subseteq \mathcal{R}^n$ is a lattice, $\Theta \subseteq \mathcal{R}^m$, and $f : X \times \Theta \to \mathcal{R}$ has **increasing differences** in (x, θ) and is **supermodular** in x, then $X^*(\theta)$ is increasing in the strong set order.*

- With vectors, we need a condition on the relationship between θ and x and also conditions on how components of x impact other components.
- What is a lattice? This is a big topic, but think of it as a set with some sense of ordering. For more, see the further study section.

1.3 Two Action Intuition

To get a handle on this stuff, consider a simple game where you face two actions: Up or Down. Your payoffs are $U(up) = \theta$ and $U(down) = 0$. Our decision to play up is just whether $\theta \geq 0$. If θ rises, we play Up more. In this sense, we have increasing differences: as θ rises, we want to play Up more, because the gains from up relative to down:

$$f(x^H, \theta^H) - f(x^L, \theta^L) = \theta^H - 0 \geq \theta^L - 0 = f(x^H, \theta^L) - f(x^L, \theta^L)$$

1.4 Example: Law of Supply from Wolitzky

Consider the classic price-taking firm's profit max problem:

$$\max_{y \in \mathcal{R}^n_+} pf(y) - q\dot{y}$$

We want to prove the law of supply, that is when the price of product rises, we produce more. let's use our new tools.

First, re-formulate variables using the dual: $x = f(y)$ and:

$$c(x) = \min_y q\dot{y}$$

s.t.

$$f(y) \geq x$$

Problem is now:

$$\max_{x \in \mathcal{R}} px - c(x)$$

that is maximize profit with respect to output, not y. First, see if this has increasing differences. Consider:

$$\left(p^H x^H - c(x^H)\right) - \left(p^H x^L - c(x^L)\right) \; ? \; \left(p^L x^H - c(x^H)\right) - \left(p^L x^L - c(x^L)\right)$$

$$p^H x^H - p^H x^L \; ? \; p^L x^H - p^L x^L$$

$$p^H (x^H - x^L) \; ? \; p^L (x^H - x^L)$$

recall that $x^H \geq x^L$ and $p^H \geq p^L$ so clearly the left is greater.

Thus we have increasing differences. Using Theorem 1, since the profit function has increasing differences, $x^*(p)$ is increasing in the strong set order. Thus we have the law of supply! (without ANY functional assumptions).

1.5 Further Study

Understanding these topics requires understanding lattices, which are sort of like ordered sets (I am not an expert). Another concept that is often applied log-supermodularity. For more on this, you can read *Supermodularity and Complementarity (1998)* by Topkis. Or just search "monotone comparative statics."